M 12

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Hwk #7

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1. (a) *Claim.* $3x$ is $O(\frac{x^2}{x+2})$.

   Proof. Consider $c = 5$ and $k = 100$. Suppose that $x \geq k = 100$. Then, for any x, $3x \leq x^2$. So, $6x \leq 2x^2$, $3x^2 + 6x \leq 5x^2$, and $3x(x+2) \leq 5x^2$. Then $3x \leq 5(\frac{x^2}{x+2}) = c(\frac{x^2}{x+2})$. So, by definition of big-O, $3x$ is $O(\frac{x^2}{x+2})$. □

   (b) *Claim.* $\frac{2x^3}{x+1}$ is not $O(x)$.

   Disproof. Suppose not, that is, suppose that $\frac{2x^3}{x+1}$ is $O(x)$. Then, consider $k = 10$ such that $x \geq k = 10$. Then $x \leq x^3$, $2x \leq 2x^3$, and $2x^2 + 2x \leq 2x^3 + 2x^2$. Factoring the left side, $2x(x+1) = 2x^3 + 2x^2$ and dividing the by the left side, we get $\frac{2x^2(x+1)}{x+1} \geq 2x$. Simplifying the fraction, $2x^2 \geq 2x = cx$. Then that means $2x \geq c$. But this fails for $x < \frac{c}{2}$. □

2. (a)

$$\begin{aligned}
f(n) =& f(n-7) + 4 \\
=& (f(n-14) + 4) + 4 \\
=& ((f(n-21) + 4) + 4) + 4 \\
=& (((f(n-28) + 4) + 4) + 4) + 4 \\
& \vdots \\
f(n) =& f(n-7i) + 4i \\
=& f(n-7i) + \sum_{k=1}^{i} 4
\end{aligned}$$

$$\begin{aligned}
n - i =& 7 \Leftrightarrow i = \frac{n}{7} \\
f(n) =& f(0) + 4(\frac{n}{7}) \\
=& 4 + 4(\frac{n}{7})
\end{aligned}$$

but because there are multiple base cases...

$$f(n) = 4 + 4\lfloor\frac{n}{7}\rfloor$$

(b)

$$
\begin{aligned}
g(n) =& g(\frac{n}{3}) + 21 \\
=& (g(\frac{n}{9}) + 21) + 21 \\
=& ((g(\frac{n}{27}) + 21) + 21) + 21 \\
=& (((g(\frac{n}{81}) + 21) + 21) + 21) + 21 \\
& \vdots \\
f(n) =& g(\frac{n}{3^i}) + \sum_{k=1}^{i} 21
\end{aligned}
$$

$$
\begin{aligned}
\frac{n}{3^i} =& 1 \Leftrightarrow \log_3 n = i \\
f(n) =& 21 * \log_3 n
\end{aligned}
$$

3. *Proof.* By strong induction on n.

Base.

$$
\begin{aligned}
f(0) &= 0 \\
&= \frac{2^0 + (-1)^{0+1}}{3} = \frac{1-1}{3} = 0 \\
f(1) &= 1 \\
&= \frac{2^1 + (-1)^{1+1}}{3} = \frac{2+1}{3} = 1 \\
f(2) &= 1 \\
&= \frac{2^2 + (-1)^{2+1}}{3} = \frac{4-1}{3} = 1
\end{aligned}
$$

Induction. Suppose that $f(k) = \frac{2^k+(-1)^{k+1}}{3}$ is true for all $1 \dots k$. Then, we need to show that $f(k+1) = \frac{2^{k+1}+(-1)^k}{3}$.

We start with the definition of the function:

$$f(k+1) = 2f(k) + f(k-1) - 2f(k-2)$$

Since we know our inductive hypothesis is true for $k \geq 1$:

$$f(k+1) = 2f(\frac{2^k + (-1)^{k+1}}{3}) + f(\frac{2^{k-1} + (-1)^k}{3}) - 2f(\frac{2^{k-2} + (-1)^{k-1}}{3})$$

The resulting equation can be simplified:

$$f(k+1) = \frac{2^{k+1}}{3} + \frac{2(-1)^{k+1}}{3} + \frac{2^{k-1}}{3} + \frac{(-1)^k}{3} + \frac{(-1)(2)(2)^{k-2}}{3} + \frac{(-1)(2)(-1)^{k-1}}{3}$$

$$3f(k+1) = 2 * 2^k + (-2)(-1)^k + \frac{1}{2}2^k + (-1)^k + -\frac{1}{2}(2)^k + 2 * (-1)^k$$

$$3f(k+1) = 2^{k+1} + (-1)^k$$

$$f(k+1) = \frac{2^{k+1} + (-1)^k}{3}$$

Which is what we needed to show. So by mathematical induction, $\forall n, f(n) = \frac{2^n+(-1)^{n+1}}{3}$. □

4. (a) *Proof.* By induction on n.

   Base. $a_2 \geq a_1$
   $a_2 = 3 - \frac{1}{1} \geq 1 = a_1$
   $2 \geq 1$ which is true.

   Induction. Suppose $a_{k+1} \geq a_k$ is true, that is $3 - \frac{1}{a_k} \geq 3 - \frac{1}{a_{k-1}}$. We need to show that $a_{k+2} \geq a_{k+1}$ is true; that $3 - \frac{1}{a_{k+1}} \geq 3 - \frac{1}{a_k}$.

   We can start with what we're trying to prove, and see if it is equivalent to a true statement.

$$a_{k+2} \geq a_{k+1} \Leftrightarrow 3 - \frac{1}{a_{k+1}} \geq 3 - \frac{1}{a_k}$$

$$\Leftrightarrow -\frac{1}{a_{k+1}} \geq -\frac{1}{a_k} \Leftrightarrow \frac{1}{a_{k+1}} \leq \frac{1}{a_k}$$

   We can substitute in for a_{k+1} and a_k using their definitions:

$$\frac{1}{3 - \frac{1}{a_k}} \leq \frac{1}{3 - \frac{1}{a_{k-1}}}$$

   And that can be simplified to:

$$3 - \frac{1}{a_k} \geq 3 - \frac{1}{a_{k-1}}$$

   By the inductive hypothesis that is true statement, and so by mathematical induction, $\forall n, a_{n+1} \geq a_n$. □

(b) *Proof 1.* By induction on n, we will prove $a_n \geq 1$ first.

Base. $a_1 = 1 \geq 1$, which is true.

Induction. Suppose that $a_k \geq 1$ for all k. Then we need to show that $a_{k+1} \geq 1$ is true, also.

From part (a), we know that $a_{k+1} \geq a_k$. We also know from the inductive hypothesis that $a_k \geq 1$ for all k. So, $a_{k+1} \geq a_k \geq 1$. Then $a_{k+1} \geq 1$, which is what we needed to show. So, by mathematical induction, $a_n \geq 1$ for all n.

Proof 2.

We know that $a_k = 3 - \frac{1}{a_{k-1}}$. Since $a_{k-1} \geq 1$ by what we proved in Proof 1, $\frac{1}{a_{k-1}} \leq 1$. Since $\frac{1}{a_{k-1}} \leq 1$ and $a_{k-1} \geq 1$, then $3 - \frac{1}{a_{k-1}}$ must be less than 3; and so $a_n < 3$ for all n. $\square$